

GENERIC RECTANGULATIONS

NATHAN READING

ABSTRACT. A rectangulation is a tiling of a rectangle by a finite number of rectangles. The rectangulation is called generic if no four of its rectangles share a single corner. We initiate the enumeration of generic rectangulations up to combinatorial equivalence by establishing an explicit bijection between generic rectangulations and a set of permutations defined by a pattern-avoidance condition analogous to the definition of the twisted Baxter permutations.

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1. INTRODUCTION

The main characters in this paper are tilings of a rectangle by finitely many rectangles. A *cross* in such a tiling is a point which is a corner of four distinct tiles. Fixing a rectangle S and considering the space of all tilings of S by n rectangles, with a uniform probability measure, the set of tilings having one or more crosses has measure zero. Thus we call a tiling *generic* if it has no crosses.

We consider generic tilings up to the natural combinatorial equivalence relation which we now describe. We orient S so that its edges are vertical and horizontal. A rectangle U in a tiling R is *below* a rectangle V if the top edge of U intersects the bottom edge of V (necessarily in a line segment rather than in a point). Similarly, U is *left of* V if the right edge of U intersects the left edge of V . A tiling R of a rectangle S is combinatorially equivalent to a tiling R' of a rectangle S' if there is a bijection from the rectangles of R to the rectangles of R' that exactly preserves the relations “below” and “left of.” A *generic rectangulation* is the equivalence class of a generic tiling. We will often blur the distinction between generic rectangulations (i.e. equivalence classes) and equivalence class representatives, in particular specifying an equivalence class by describing a specific tiling.

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n	Generic rectangulations	n	Generic rectangulations
1	1	15	58,457,558,394
2	2	16	533,530,004,810
3	6	17	4,970,471,875,914
4	24	18	47,169,234,466,788
5	116	19	455,170,730,152,340
6	642	20	4,459,456,443,328,824
7	3,938	21	44,300,299,824,885,392
8	26,194	22	445,703,524,836,260,400
9	186,042	23	4,536,891,586,511,660,256
10	1,395,008	24	46,682,404,846,719,083,048
11	10,948,768	25	485,158,560,873,624,409,904
12	89,346,128	26	5,089,092,437,784,870,584,576
13	754,062,288	27	53,845,049,871,942,333,501,408
14	6,553,942,722		

TABLE 1. The number of generic rectangulations with n rectangles

Our main result is a bijection between generic rectangulations with n rectangles and a class of permutations in S_n that we call *2-clumped permutations*. These are the permutations that avoid the patterns 3-51-2-4, 3-51-4-2, 2-4-51-3, and 4-2-51-3, in the notation of Babson and Steingrímsson [5], which is explained in Section 2. The author's counts of generic rectangulations, for small n , are shown in Table 1.

We define k -clumped permutations in Section 2. For now, to place the 2-clumped permutations in context, we note that the 1-clumped permutations are the *twisted Baxter permutations*, which are in bijection with the better-known *Baxter permutations*. Baxter permutations are also relevant to the combinatorics of rectangulations. Indeed, Baxter permutations are in bijection [2, 19] with the *mosaic floorplans* considered in the VLSI (Very Large Scale Integration) circuit design literature [13]. Mosaic floorplans are certain equivalence classes of generic rectangulations. (A similar result linking equivalence classes of generic rectangulations to pattern-avoiding permutations is given in [4].) In light of results of [1], the bijection from Baxter permutations to mosaic floorplans can be rephrased as a bijection to a subclass of the generic rectangulations that we call *diagonal rectangulations*, which figure prominently in this paper.

The symbol G_n will denote the set of 2-clumped permutations. Let gRec_n be the set of generic rectangulations with n rectangles. The bijection from G_n to gRec_n is defined as the restriction of a map $\gamma : S_n \rightarrow \text{gRec}_n$. We show that γ is surjective and that its fibers are the congruence classes of a lattice congruence on the weak order on S_n . We do not prove directly that the fibers of γ define a congruence. Instead, we recognize the fibers as the classes of a congruence arising as one case of a construction from [17], where lattice congruences on the weak order are used to construct sub Hopf algebras of the Malvenuto-Reutenauer Hopf algebra of permutations. The results of [17] show that the 2-clumped permutations are a set of congruence class representatives. Thus the restriction of γ is a bijection from G_n to gRec_n .

Note added in proof. After this paper was accepted, the author became aware of a substantial literature studying generic rectangulations under the name *rectangular drawings*. This literature includes some results on asymptotic enumeration as well as computations of the exact cardinality of gRec_n for many values of n . See, for example, [3, 11, 14]. In particular, the main result of this paper answers an open question posed in [3, Section 5].

2. CLUMPED PERMUTATIONS

In this section, we define k -clumped permutations. We begin with a review of generalized pattern avoidance in the sense of Babson and Steingrímsson [5]. Let $y = y_1 \cdots y_k \in S_k$, and let \tilde{y} be a word created by inserting a dash between some letters of $y_1 \cdots y_k$, with at most one dash between each adjacent pair. A subsequence $x_{i_1} \cdots x_{i_k}$ of $x_1 \cdots x_n$ is an *occurrence* of the pattern \tilde{y} in a permutation $x \in S_n$ if the following two conditions are satisfied: First, for all $j, l \in [k]$ with $j < l$, the inequality $x_{i_j} < x_{i_l}$ holds if and only if $y_j < y_l$ holds. Second, if y_j and y_{j+1} are not separated by a dash in \tilde{y} , then $i_j = i_{j+1} - 1$. That is, the dashes indicate which elements of the subsequence are not required to be adjacent in x . For example, the subsequence 4512 of $45312 \in S_5$ is an occurrence of the pattern 3-4-1-2, or an occurrence of the pattern 34-12, but not an occurrence of the pattern 3-41-2. If there is no occurrence of the pattern \tilde{y} in x , then we say that x *avoids* \tilde{y} .

To define k -clumped permutations, we first consider the *twisted Baxter permutations*, defined in [17] and shown in unpublished notes by West [18] to be in bijection with Baxter permutations. A published proof can be found in [15] or [12]. The twisted Baxter permutations are the permutations that avoid the patterns 2-41-3 and 3-41-2. This pattern-avoidance condition on a permutation $x = x_1 \cdots x_n$ can be rephrased as follows: For every descent $x_i > x_{i+1}$, the values strictly between x_{i+1} and x_i are either all to the left of x_i or all to the right of x_{i+1} . (The Baxter permutations are defined by a similar condition: They are the permutations avoiding 3-14-2 and 2-41-3.)

In any permutation x , we define a *clump* associated to a descent $x_i > x_{i+1}$ to be a nonempty maximal sequence of consecutive values strictly between x_i and x_{i+1} , all of which are on the same side of the entries $x_i x_{i+1}$. No requirement is made on the positions, relative to each other, of the values in the clump. For example, in the permutation $269153847 \in S_9$, there are four clumps associated to the descent $9 > 1$, namely 2, 345, 6, and 78.

The pattern avoidance condition defining twisted Baxter permutations is that each descent $x_i > x_{i+1}$ has at most one clump, so we refer to twisted Baxter permutations as 1-clumped permutations. More generally, a k -clumped permutation is a permutation x such that each descent $x_i > x_{i+1}$ has at most k clumps. One can easily rephrase the definition of k -clumped permutations in terms of generalized patterns avoidance (avoiding $2 \binom{k}{2}! \left(\frac{k}{2} + 1\right)!$ generalized patterns if k is even or $2 \left(\frac{k+1}{2}\right)! \left(\frac{k+1}{2}\right)!$ generalized patterns if k is odd). For example, the 2-clumped permutations, which play the central role in this paper, are the permutations avoiding 3-51-2-4, 3-51-4-2, 2-4-51-3, and 4-2-51-3. By convention, the only (-1) -clumped permutation is the identity. The 0-clumped permutations are the permutations such that if $x_i > x_{i+1}$ then $x_i - 1 = x_{i+1}$. Equivalently, they are the permutations avoiding 31-2 and 2-31. These permutations in S_n are in bijection with subsets of $\{1, 2, \dots, n-1\}$. The 3-clumped permutations appear not to have been considered

before. For n from 1 to 9, the numbers of 3-clumped permutations are 1, 2, 6, 24, 120, 712, 4804, 35676 and 284816.

The *weak order* on S_k is a lattice whose cover relations are $x < y$ with $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_k$ such that $x_i = y_{i+1} < y_i = x_{i+1}$ for some $i \in [k-1]$, with $x_j = y_j$ for $j \notin \{i, i+1\}$. A *join-irreducible* permutation is a permutation $x \in S_k$ with exactly one descent, meaning that, for some index $i \in [k-1]$, we have $x_i > x_{i+1}$ but $x_j < x_{j+1}$ for every $j \in [k-1]$ with $j \neq i$. (Such a permutation is join-irreducible in the weak order in the usual lattice-theoretic sense.)

We now review a construction from [17, Section 9]. A join-irreducible element $x \in S_k$ is called *untranslated* if its unique descent $x_i > x_{i+1}$ has $x_i = k$ and $x_{i+1} = 1$. In this case, a *scramble* of x is any permutation y such that $y_i = k$, $y_{i+1} = 1$ and every entry j with $1 < j < k$ occurs to the left of position i in x if and only if it occurs to the left of position i in y . Let y be a scramble of x and let \tilde{y} be obtained from y by inserting a dash between each pair of consecutive entries except between k and 1. We say that the scramble y of x occurs *with adjacent cliff* if the pattern \tilde{y} occurs.

Let C be any collection of untranslated join-irreducible elements in S_k , with k varying, so that, for example, C may be $\{312, 2413\}$. The following is essentially [17, Theorem 9.3].

Theorem 2.1. *For each n , there exists a unique congruence $\mathcal{H}(C)_n$ on the weak order on S_n with the following properties:*

- (i) *A permutation z is the minimal element in its $\mathcal{H}(C)_n$ -class if and only if, for every $x \in C$ and all scrambles y of x , the permutation z avoids occurrences of y with adjacent cliff.*
- (ii) *Suppose $w < z$ in the weak order, and let z_i and z_{i+1} be the adjacent entries of z that are swapped to convert z to w , with $z_i > z_{i+1}$. Then $w \equiv z$ modulo $\mathcal{H}(C)_n$ if and only if there exists $x \in C$, a scramble $y \in S_k$ of x , and an occurrence of \tilde{y} in z such that the entry of z corresponding to the entry k in \tilde{y} is z_i and the entry of z corresponding to 1 in y is z_{i+1} .*

In [17], the congruence $\mathcal{H}(C)_n$ is constructed for the purpose of building combinatorial Hopf algebras. Here, we can take Theorem 2.1 as the definition of $\mathcal{H}(C)_n$. Property (i) in Theorem 2.1 is a direct restatement of [17, Theorem 9.3], while property (ii) is the key point in the proof of [17, Theorem 9.3]. It is easy and well-known that in a congruence on a finite lattice, each congruence class is an interval. Thus a congruence is uniquely determined by the set of cover relations $w < z$ such that $w \equiv z$. Furthermore, the minimal permutations described in Property (i) are a system of congruence class representatives.

Let Γ be the congruence $\mathcal{H}(\{35124, 24513\})_n$ on S_n . Theorem 2.1 specializes to the following:

Proposition 2.2.

- (1) *A permutation is the minimal element in its Γ -class if and only if it is a 2-clumped permutation.*
- (2) *Suppose $x < y$ in the weak order, and let e and a be the adjacent entries that are swapped to convert y to x , with $a < e$. Then $x \equiv y$ modulo Γ if and only if there are entries b, c , and d in y with $a < b < c < d < e$ such that b and d are on the same side of ea , while c is on the other side of ea .*

More generally, for each $k \geq -1$, there is a congruence described by Theorem 2.1 such that the minimal elements of congruence classes are exactly the k -clumped permutations.

3. THE MAP FROM PERMUTATIONS TO GENERIC RECTANGULATIONS

In this section, we define a map γ from S_n to gRec_n . We will see, in Section 4, that γ restricts to a bijection from the set of 2-clumped permutations to gRec_n . The key point in the proof that γ restricts to a bijection will be the fact that its fibers are the congruence classes of the congruence Γ defined at the end of Section 2.

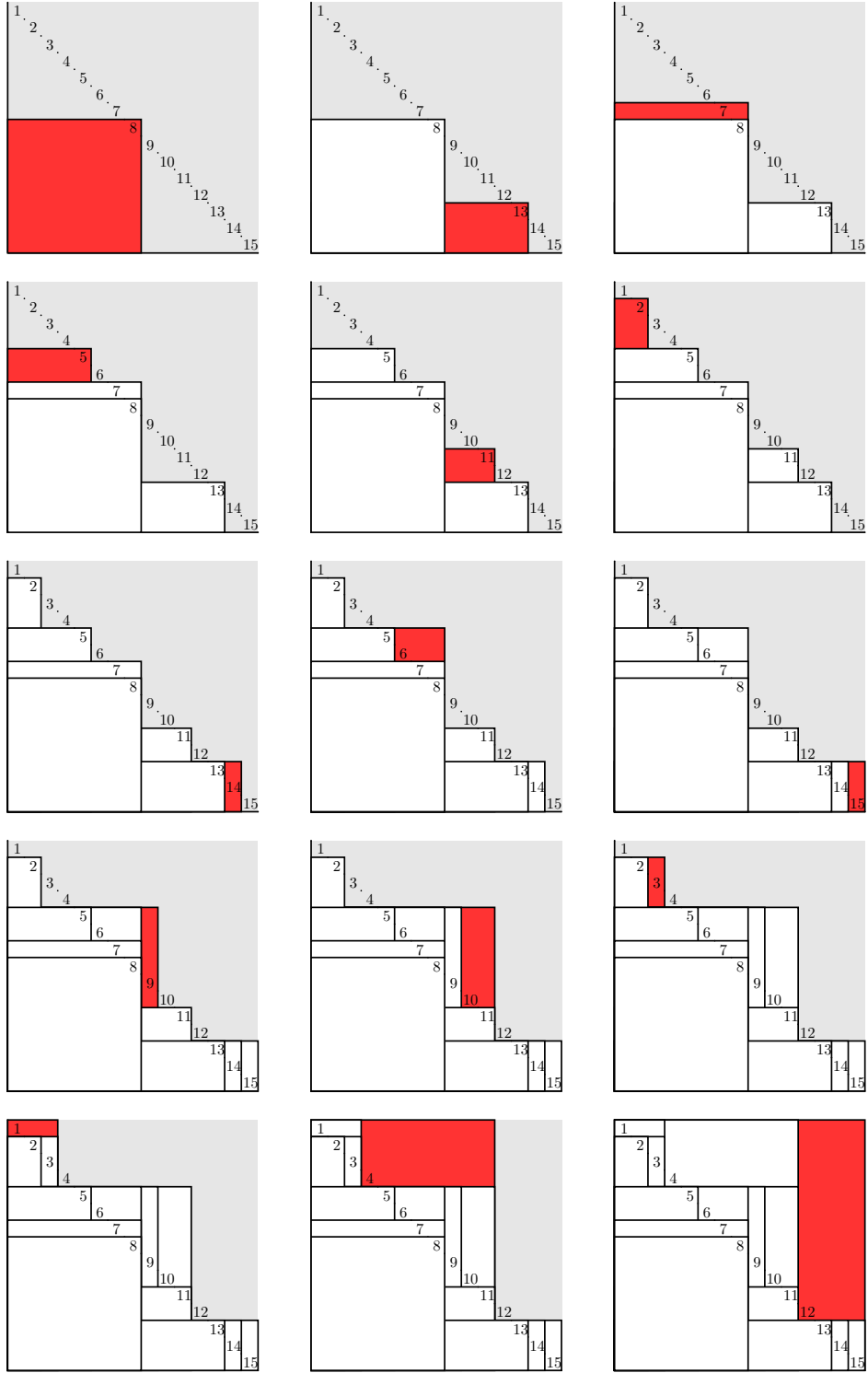
To define the map γ , we first consider a smaller class of rectangulations which we call *diagonal rectangulations* and a map from permutations to diagonal rectangulations. The *diagonal* of the underlying rectangle S is the line segment connecting the top-left corner of S to the bottom-right corner of S . Recall that each rectangulation is a combinatorial equivalence class. A rectangulation is a diagonal rectangulation if it has a representative in which each rectangle's interior intersects the diagonal. A diagonal rectangulation is in particular a generic rectangulation, because if any four rectangles have a common vertex, it is impossible for all of their interiors to intersect the diagonal. Diagonal rectangulations have been considered under other names, for example in [1, 9, 10].

We now review, from [15], the definition of a map ρ from permutations to diagonal rectangulations. Maps closely related to ρ have appeared prior to [15], for example in [1, 10]. To define ρ , first draw $n + 1$ distinct *diagonal points* on the diagonal of S , with one of the points being the top-left corner of S and another being the bottom-right corner of S . Number the spaces between the diagonal points as $1, 2, \dots, n$, from top-left to bottom-right. Given $x \in S_n$, read the sequence $x_1 \cdots x_n$ from left to right and draw a rectangle for each entry according to the following recursive procedure:

Let T be the union of the left and bottom edges of S with the rectangles drawn in the first $i - 1$ steps of the construction. It will be apparent by induction that T is left- and bottom-justified. To draw the i^{th} rectangle, consider the label x_i on the diagonal. If the diagonal point p immediately above/left of the label x_i is not in T , then the top-left corner of the new rectangle is the rightmost point of T that is directly left of p . If p is in T (necessarily on the boundary of T), then the top-left corner of the new rectangle is the highest point of T directly above p . If the diagonal point p' immediately below/right of the label x_i is not in T , then the bottom-right corner of the new rectangle is the highest point of T that is directly below p' . If p' is in T then the bottom-right corner of the new rectangle is the rightmost point of T that is directly to the right of p' .

Example 3.1. Figure 1 illustrates the map ρ . In each step, the new rectangle is shown in red (the darker gray when not viewed in color), and the set T consists of the white rectangles together with the left and bottom edges of S . The part of S not covered by rectangles is shaded in light gray.

Given a diagonal rectangulation R , we number the rectangles in R according to the position of their intersections with the diagonal, starting with rectangle 1, which contains the top-left corner of S and ending at rectangle n , which contains the bottom-right corner of S . Thus, for example, in constructing the rectangulation $\rho(x)$, we first construct the rectangle numbered x_1 , then the rectangle numbered x_2 ,

FIGURE 1. Steps in the construction of $\rho(8\textcircled{13}75\textcircled{11}2\textcircled{14}6\textcircled{15}9\textcircled{10}314\textcircled{12})$

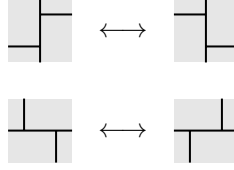


FIGURE 2. Wall slides

etc. Say a permutation $x = x_1 \cdots x_n$ is *compatible with* a diagonal rectangulation R if and only if, for every $i \in [n]$, the left and bottom edges of the rectangle numbered x_i are contained in the union of the left and bottom edges of S with the rectangles numbered x_1, \dots, x_{i-1} . Equivalently, x is compatible with R if, for every $i \in [n]$, the union of the rectangles numbered x_1, \dots, x_i is left- and bottom-justified. The following fact is established in the proof [15, Proposition 6.2], which asserts that ρ is surjective.

Proposition 3.2. *Given a diagonal rectangulation R , the fiber $\rho^{-1}(R)$ is the set of permutations in S_n that are compatible with R .*

The following proposition, which is the concatenation of [15, Proposition 4.5] and [15, Theorem 6.3], shows in particular that the fibers of ρ constitute a congruence of the kind described in Theorem 2.1.

Proposition 3.3. *Suppose $x \leq y$ in the weak order, and let d and a be the adjacent entries that are swapped to convert y to x , with $a < d$. Then $\rho(x) = \rho(y)$ if and only if there are entries b and c , with $a < b < c < d$, such that b and c are on opposite sides of da in y .*

In some of the literature on floorplanning for integrated circuits, generic rectangulations are referred to as *mosaic floorplans*, but in that literature, the term mosaic floorplan always implies a coarser equivalence relation than the combinatorial equivalence used to define rectangulations as equivalence classes. Specifically, two generic rectangulations are equivalent as mosaic floorplans if and only if they are related by a sequence of what we call *wall slides*. A *wall* in a rectangulation R is a line segment in the underlying rectangle S , not contained in an edge of S , that is maximal with respect to the property of not intersecting the interior of any rectangle of R . A wall slide along a wall W is the operation taking two walls of R that end in W , from opposite sides, and sliding them past each other, without changing any of the other incidences in R . Wall slides come in two orientations, as illustrated in Figure 2. The following is a very special case of [1, Theorem 4].

Proposition 3.4. *Given a generic rectangulation R , there exists a unique diagonal rectangulation R' such that R and R' are equivalent as mosaic floorplans.*

To see Proposition 3.4 as a special case of [1, Theorem 4], we need the definition of a diagonal rectangulation given in [15, Section 5]: Let X be a set of $n - 1$ distinct points on the diagonal of S , none of which is the top-left corner or bottom-right corner of S . Then a diagonal rectangulation of (S, X) is a generic rectangulation such that every wall contains a point of X and such that every point of X lies on a wall. By [15, Proposition 5.2], this definition is equivalent to the earlier definition.

Suppose R is a generic rectangulation and let R' be the diagonal rectangulation that is equivalent to R as a mosaic floorplan. As before, number the rectangles in

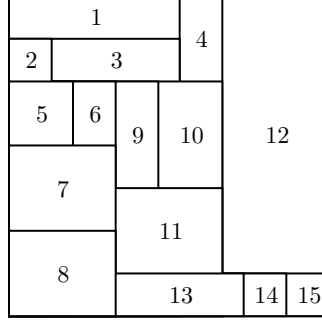


FIGURE 3. A generic rectangulation

R' according to the position of their intersections with the diagonal, 1 to n from top-left to bottom-right. Letting this numbering propagate along wall slides in the obvious way, we obtain a numbering of the rectangles of R . For each vertical wall W of R , we produce a permutation σ_W of a subset of $[n]$ as follows: Moving from the bottom endpoint of W to the top endpoint of W , when we come to a wall W' that is incident to W on the left, we record the number of the rectangle that has its right edge in W and its *bottom* edge in W' . When we come to a wall W' that is incident to W on the right, we record the number of the rectangle that has its left edge in W and its *top* edge in W' . The resulting partial permutation σ_W is called the *wall shuffle* associated to W , because it is obtained by shuffling two sequences: the decreasing sequence of numbers of rectangles whose right edge is contained in W (excluding the bottom such rectangle) from bottom to top and the decreasing sequence of numbers of rectangles whose left edge is contained in W (excluding the top such rectangle) from bottom to top.

For each horizontal wall W , we construct the wall shuffle associated to W in a similar manner. Moving from the left endpoint of W to the right endpoint of W , when we come to a wall W' that is incident to W on the top, we record the number of the rectangle that has its bottom edge in W and its *right* edge in W' . When we come to a wall W' that is incident to W on the bottom, we record the number of the rectangle that has its top edge in W and its *left* edge in W' . The partial permutation σ_W , in this case, is obtained by shuffling two increasing sequences: the sequence of numbers of rectangles whose bottom edge is contained in W (excluding the rightmost such rectangle) from left to right and the sequence of numbers of rectangles whose top edge is contained in W (excluding the leftmost such rectangle) from left to right.

Example 3.5. Figure 3 shows a generic rectangulation R whose associated diagonal rectangulation R' is the rectangulation from Figure 1. The numbering of rectangles is inherited from R' . Tables 2 and 3 show the wall shuffles associated to R .

Specifying a generic representation R is equivalent to specifying the associated diagonal rectangulation R' along with the wall shuffles for each wall. For some walls, there may be only one shuffle possible, and this unique shuffle may be empty. The shuffles may be chosen arbitrarily (among shuffles of the appropriate rectangle numbers) and independently for each wall, and each sequence of choices of R' and the wall shuffles yields a different generic rectangulation.

Rectangles left of wall	Rectangles right of wall	Wall shuffle
2	3	<i>empty</i>
5	6	<i>empty</i>
8, 7, 6	13, 11, 9	$\textcircled{13}7\textcircled{11}6$
9	10	<i>empty</i>
3, 1	4	1
11, 10, 4	12	$\textcircled{10}4$
13	14	<i>empty</i>
14	15	<i>empty</i>

TABLE 2. Wall shuffles in vertical walls of the rectangulation of Figure 3

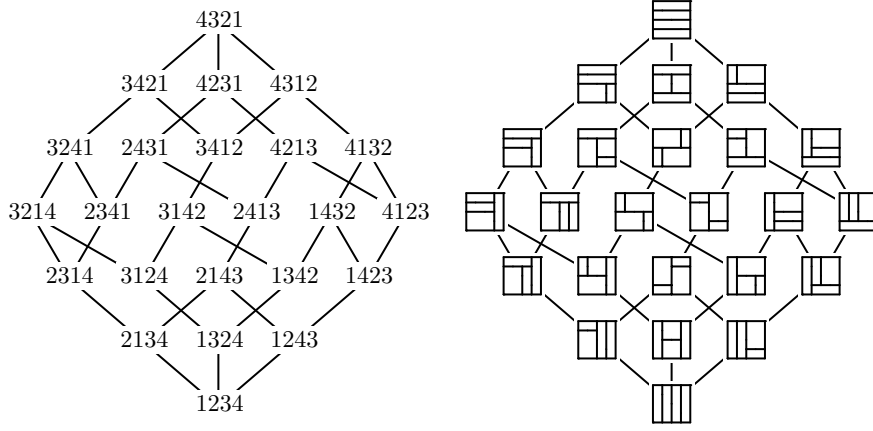
Rectangles above wall	Rectangles below wall	Wall shuffle
1	2,3	3
2, 3, 4	5, 6, 9, 10	269 $\textcircled{10}$ 3
5, 6	7	5
9, 10	11	9
7	8	<i>empty</i>
11, 12	13, 14, 15	$\textcircled{11}\textcircled{14}\textcircled{15}$

TABLE 3. Wall shuffles in horizontal walls of the rectangulation of Figure 3

When a wall slide is performed along a wall W , the move alters σ_W by swapping two adjacent entries which number rectangles on opposite sides of W . Since a wall slide only changes the combinatorics locally, performing a wall slide along W does not alter the wall shuffle for any other wall.

We now define the map $\gamma : S_n \rightarrow \text{gRec}_n$. Let $x = x_1x_2 \cdots x_n \in S_n$ and construct $R' = \rho(x)$. Let W be a vertical wall in R' and consider the rectangles in R' having their right edges contained in W . By construction, the numbers of these rectangles form a decreasing subsequence of $x_1x_2 \cdots x_n$. Similarly, the numbers of the rectangles in R' having their left edges contained in W are a decreasing subsequence of $x_1x_2 \cdots x_n$. Thus we can specify a wall shuffle σ_W by taking the subsequence of $x_1x_2 \cdots x_n$ consisting of the appropriate rectangle numbers. For a horizontal wall W , the numbers of the rectangles having their top edges contained in W form an increasing subsequence of $x_1x_2 \cdots x_n$ and the numbers of the rectangles having their bottom edges contained in W form an increasing subsequence of $x_1x_2 \cdots x_n$, so, in this case as well, we can specify a wall shuffle for W by taking an appropriate subsequence of $x_1x_2 \cdots x_n$. The diagonal rectangulation R' together with all of these wall shuffles define the generic rectangulation $\gamma(x)$.

Example 3.6. This is a continuation of Examples 3.1 and 3.5. Figure 1 shows the construction of $\rho(x)$ for $x = 8\textcircled{13}75\textcircled{11}2\textcircled{14}6\textcircled{15}9\textcircled{10}314\textcircled{12}$. To construct $\gamma(x)$, we look at each wall of $\rho(x)$. For example, $\rho(x)$ has a horizontal wall W with rectangles 2, 3, and 4 above W and rectangles 5, 6, 9, and 10 below W . The restriction of x to the set $\{2, 3, 6, 9, 10\}$ is 269 $\textcircled{10}$ 3. Thus $\gamma(x)$ is a rectangulation that is mosaic equivalent to $\rho(x)$ and that has a wall shuffle 269 $\textcircled{10}$ 3. Considering similarly the

FIGURE 4. $\gamma : S_4 \rightarrow \text{gRec}_4$

other five horizontal walls of $\rho(x)$ and the eight vertical walls of $\rho(x)$, we see that $\gamma(x)$ is the rectangulation shown in Figure 3. (Cf. Tables 2 and 3.)

Example 3.7. Figure 4 shows the map γ applied to every permutation in S_4 . The permutations in S_4 are shown in the weak order, and the 24 rectangulations in gRec_4 are shown in the corresponding order. As a byproduct of the results of Section 4, the map $\gamma : S_n \rightarrow \text{gRec}_n$ induces a lattice structure on gRec_n such that γ is a surjective lattice homomorphism.

4. MAIN THEOREM

In this section, we prove our main theorem.

Theorem 4.1. *The restriction of γ is a bijection from the set of 2-clumped permutations in S_n to the set of generic rectangulations with n rectangles.*

The proof of Theorem 4.1 is accomplished by proving three propositions.

Proposition 4.2. *The map $\gamma : S_n \rightarrow \text{gRec}_n$ is surjective.*

Proof. Let R' be any diagonal rectangulation and choose an arbitrary wall shuffle for each wall of R' . We need to show that there exists $x = x_1x_2 \cdots x_n \in S_n$ such that $\rho(x) = R'$ and such that each chosen wall shuffle is a subsequence of $x_1x_2 \cdots x_n$. That is, we need to show that the rectangles of R' can be ordered consistent with the requirements of Proposition 3.2 and with the wall shuffles.

Suppose, for $1 \leq i \leq n$, that we have chosen $i-1$ rectangles in an order consistent with the requirements of Proposition 3.2 and with the wall shuffles. We will show that we can choose a rectangle in step i that also satisfies the requirements. Since we have chosen consistent with Proposition 3.2, the union T of the $i-1$ rectangles chosen with the left and bottom edges of S , is a left- and bottom-justified set. To satisfy the requirement of Proposition 3.2 in step i , we must choose a rectangle whose bottom and left edges are contained in T . To show that we can choose such a rectangle consistent with the wall shuffles, we extend an argument from the proof of [15, Proposition 6.2].

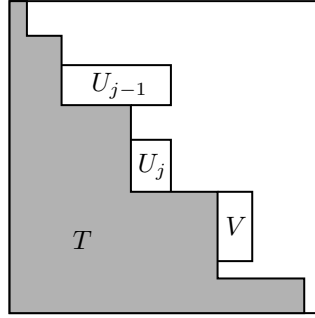


FIGURE 5. A figure illustrating the proof of Proposition 4.2

The top-right boundary of T is a polygonal path from the top-left corner of S to the bottom-right corner of S , always moving directly right or directly down. Each point where the path turns from moving down to moving right is the bottom-left corner of a rectangle of R' that is not contained in T . We index these rectangles U_1, \dots, U_m from top-left to bottom-right. The left edge of U_1 is necessarily contained in T , or else we were wrong to index it as U_1 . Thus if U_1 fails to have both its bottom and left edges in T , then its bottom edge is not contained in T . This implies that the left edge of U_2 is contained in T . We continue until we find the first j such that the bottom edge of U_j is contained in T . Since the bottom edge of U_m is in T , such a j exists. Necessarily, the left edge of U_j is also contained in T .

We now consider the walls containing the edges of U_j . First, let W_l be the wall containing the left edge of U_j . (If $j = 1$ and the left edge of U_j is in the left edge of S , then there is no wall shuffle associated to the left edge of U_j .) Because the bottom edge of U_{j-1} is not contained in T , the top endpoint of W_l is contained in the bottom edge of U_{j-1} , as illustrated in Figure 5. (If $j = 1$ and the left edge of U_j is not in the left edge of S , then the top vertex of W_l is in the top edge of S .) We conclude that all rectangles adjacent to and left of W_l are contained in T . Thus we can pick U_j in step i consistent with the wall shuffle in W_l .

Second, let W_t be the wall containing the top edge of U_j . If W_t is also the wall containing the bottom edge of U_{j-1} , then since the bottom edge of U_{j-1} is not contained in T , the top-left corner p of U_j is the top-right corner of another rectangle of R' . Since that other rectangle must intersect the diagonal, every point on W_t from p rightwards is above the diagonal. Thus U_{j-1} is the rightmost of the rectangles adjacent to and above W_t , because otherwise the bottom-right corner of U_{j-1} is the bottom-left corner of a rectangle of R' that doesn't intersect the diagonal. Since the left edge of U_{j-1} is contained in T , all other rectangles adjacent to and above W_t are constructed in steps 1 through $i - 1$. Thus we can pick U_j in step i consistent with the wall shuffle in W_t . If W_t is not the wall containing the bottom edge of U_{j-1} , then the left endpoint of W_t is also the top-left corner of U_j . (This is the case that is illustrated in Figure 5.) In this case, U_j is leftmost among rectangles adjacent to and below W_t , so U_j does not figure in the wall shuffle in W_t .

We have shown that picking U_j in step i is allowed by Proposition 3.2 and by the wall shuffles in the walls W_l and W_t . Let W_r be the wall containing the right edge of U_j and let W_b be the wall containing the bottom edge of U_j , if these exist. If $j = m$, then there is no wall W_r and either there is no wall W_b or U_j is the

rightmost rectangle adjacent to and above W_b , so that U_j does not figure in the wall shuffle in W_b . Thus if $j = m$, the rectangle U_j can be picked in step i . If, on the other hand, $j < m$, then U_j can be picked in step i if and only if picking it is allowed by the wall shuffle in W_r and by the wall shuffle in W_b .

Let W'_l be the wall containing the left edge of U_{j+1} and let W'_t be the wall containing the top edge of U_{j+1} . We will prove the following claim: If picking U_j in step i is disallowed by the wall shuffle in W_r , or if it is disallowed by the wall shuffle in W_b , then the left edge of U_{j+1} is contained in T , and picking U_{j+1} in step i is allowed by the wall shuffle in W'_l and by the wall shuffle in W'_t .

First, suppose that picking U_j in step i is disallowed by the wall shuffle in W_r . If the bottom endpoint of W_r is also the bottom-right corner of U_j (as shown in Figure 5), then U_j is the lowest of the rectangles adjacent to and left of W_r . This would contradict the supposition that picking U_j next is disallowed by the wall shuffle in W_r , so we conclude that the wall W_r continues below U_j . Since the bottom edge of U_j is contained in T , it follows that the bottom right corner of U_j is the next convex corner of T . In particular, the wall shuffle in W_r requires us to choose U_{j+1} before U_j . Since the wall W'_l coincides with W_r , we know that the wall shuffle in W'_l does not prevent us from choosing U_{j+1} next. Also, we see that the left edge of U_{j+1} is contained in T : Otherwise the right edge of U_j intersects the left edge of U_{j+1} , making U_{j+1} the topmost of the rectangles adjacent to and right of W_r (because R' is a diagonal rectangulation). This contradicts the supposition that picking U_j next is disallowed by the wall shuffle in W_r . Furthermore, we see that the top-left corner of U_{j+1} is strictly below the convex corner of T separating U_j from U_{j+1} : Otherwise, that convex corner is the corner of four rectangles of R' (including U_j and U_{j+1}). Thus U_{j+1} is the leftmost rectangle adjacent to and below W'_t , so it does not figure in the wall shuffle in W'_t .

Next, suppose that picking U_j in step i is disallowed by the wall shuffle in W_b . Let V be leftmost among rectangles adjacent to and below W_b that are not contained in T . The rectangle V is also shown in Figure 5. Since picking U_j in step i is disallowed by the wall shuffle in W_b , the rectangle V exists and comes before U_j in the wall shuffle for W_b . The top endpoint of the wall W'_l is the top-left corner of V , and in particular is contained in T . Thus all of the rectangles adjacent to and left of W'_l are in T , so that choosing U_{j+1} in step i is allowed by the wall shuffle in W'_l . If $V = U_{j+1}$, then $W'_t = W_b$, and we already know that the wall shuffle in W_b requires us to pick V next. If V is not U_{j+1} , then the left endpoint of W'_t is contained in T , so U_{j+1} is the leftmost rectangle adjacent to and below W'_t , and thus U_{j+1} does not figure in the wall shuffle in W'_t . In either case, the left edge of U_{j+1} is contained in T , and we have proved the claim.

If the bottom edge of U_{j+1} is contained in T , then the claim implies that picking U_{j+1} in step i is allowed by Proposition 3.2 and by the wall shuffles in the walls W'_l and W'_t . We can thus argue for U_{j+1} just as we have argued above for U_j . If the bottom edge of U_{j+1} is not contained in T , then we find the first $k > j$ such that the bottom edge of U_k is contained in T , and start over as above, replacing U_j with U_k . Eventually, we will find a rectangle that can be picked, because the bottom edge of U_m is contained in T and because, as mentioned above, wall shuffles in the walls below U_m and to the right of U_m will never prevent its being picked. \square

Proposition 4.3. *Suppose $x \prec y$ in the weak order, and let e and a be the adjacent entries that are swapped to convert y to x , with $a < e$. Then $\gamma(x) = \gamma(y)$ if and*

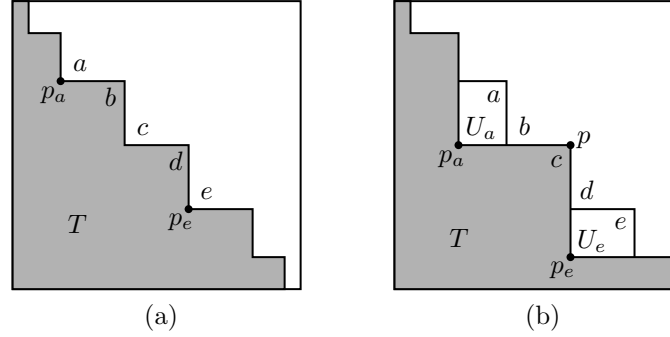


FIGURE 6. Figures illustrating the proof of Proposition 4.3

only if there are entries b , c , and d in y , with $a < b < c < d < e$, such that b and d are on the same side of ea , while c is on the other side of ea .

Proof. Both conditions in the proposition imply that $\rho(x) = \rho(y)$, by the definition of γ and by Proposition 3.2. Throughout the proof, let R' be the diagonal rectangulation $\rho(x) = \rho(y)$. We claim that $\gamma(x) = \gamma(y)$ if and only if U_a and U_e are not adjacent to any common wall of R' . Indeed, if U_a and U_e are not adjacent to any common wall of R' , then x and y must define the same wall shuffles on the walls of R' , so $\gamma(x) = \gamma(y)$. Conversely, suppose U_a and U_e are adjacent to a common wall W of R' . The assumption that $\rho(x) = \rho(y)$ rules out the possibility that U_a and U_e are both on the same side of W , so U_a and U_e are on opposite sides of W . Since U_a is chosen immediately before U_e when R' is constructed as $\rho(x)$ but immediately after U_e when constructing R' as $\rho(y)$, we see that a and e are both entries in σ_W . That is, if W is vertical, then neither U_a nor U_e is the bottom-most rectangle adjacent to W on the left, and neither U_a nor U_e is the topmost rectangle adjacent to W on the right. Similarly, if W is horizontal, neither of the two rectangles are the leftmost rectangle below W nor the rightmost rectangle above W . We conclude that $\gamma(x) \neq \gamma(y)$ and we have proved the claim.

Suppose there are entries b , c , and d in y , with $a < b < c < d < e$, such that b and d precede ea , but c follows ea . Let T be the union of the left and bottom edges of S with the rectangles chosen before U_a and U_e when R' is constructed as $\rho(x)$ or $\rho(y)$. In the construction of R' as $\rho(x)$, the rectangle U_a is chosen next, but in the construction of R' as $\rho(y)$, the rectangle U_e is chosen next. Thus both $T \cup U_a$ and $T \cup U_e$ are bottom- and left-justified sets. Therefore, every point of U_e is strictly below and strictly to the right of every point of U_a . See Figure 6.a, ignoring, for now, the labels p_a and p_e . In particular, U_a and U_e are not adjacent to a common wall of R' . By the claim, $\gamma(x) = \gamma(y)$.

Similarly, if there are entries b , c , and d in y , with $a < b < c < d < e$, such that c precedes ea , but b and d follow ea , we see that every point of U_e is strictly below and strictly to the right of every point of U_a . See Figure 6.b, ignoring the labels p , p_a , and p_e . In particular, U_a and U_e are not adjacent to a common wall of R' , so $\gamma(x) = \gamma(y)$.

Conversely, suppose $\gamma(x) = \gamma(y)$. Let T be as above. For every concave corner p of the top-right boundary of T , there is a rectangle of U whose bottom-left corner is p . Let p_a be the bottom-left corner of U_a and let p_e be the bottom-left corner

of U_e . Both p_a and p_e are concave corners of the boundary of T . There are two possibilities: The first is that, looking from top-left to bottom-right at the concave corners of the boundary of T , there is some concave corner between p_a and p_e . In this case, there exist entries b , c , and d in y , with $a < b < c < d < e$, such that b and d are before ea , while c is after ea , as illustrated by Figure 6.a. The second possibility is that, from top-left to bottom-right, there are no other concave corners between p_a and p_e . Thus there is a single convex corner p of T between p_a and p_e . Since R' equals both $\rho(x)$ and $\rho(y)$, both U_a and U_e have their bottom and left edges contained in T . If p is the bottom-right corner of U_a , or if p is the top-right corner of U_e , then U_a and U_e are adjacent to a common wall. By the claim, this is a contradiction to the supposition that $\gamma(x) = \gamma(y)$. Thus there exist entries b , c , and d in y , with $a < b < c < d < e$, such that c is before ea , while b and d are after ea , as illustrated in Figure 6.b. \square

Proposition 4.2 asserts that every fiber of γ is nonempty. The following proposition characterizes the fibers more exactly, and completes the proof of Theorem 4.1.

Proposition 4.4. *Each fiber of γ is a Γ -class. In particular, each fiber of γ contains a unique 2-clumped permutation.*

Proof. By Proposition 2.2.2 and Proposition 4.3, the fibers of γ are unions of Γ -classes. Suppose x and y are distinct permutations with $\gamma(x) = \gamma(y)$. Then $\rho(x) = \rho(y)$ and x and y are consistent with the same set of wall shuffles. We will show that x and y are congruent modulo Γ . Let i be the smallest index such that $x_i \neq y_i$. We argue by induction on $n - i$.

There is some $k > i$ such that $y_k = x_i$. Since $\gamma(x) = \gamma(y)$, either the rectangle numbered x_i or the sequence of rectangles numbered $y_i y_{i+1} \cdots y_k$ can be chosen next, consistent with the requirements of Proposition 3.2 and with the wall shuffles. We conclude that the entry $y_k = x_i$ does not participate in any wall shuffles with any of the entries $y_i y_{i+1} \cdots y_{k-1}$.

Consider the sequence of permutations starting with y and moving the entry y_k to the left one place at a time, without changing the relative positions of the other entries, with the final entry y' in the sequence having y_k in position i . Then γ is constant on the sequence. Since each pair of adjacent permutations in the sequence is a covering pair in the weak order, each pair is related as described in Proposition 4.3. But then Proposition 2.2.2 says that the entire sequence is contained in one Γ -class, so that in particular y' and y are congruent modulo Γ . Since $\gamma(x) = \gamma(y')$ and x and y' agree in positions 1 through i , by induction we conclude that x and y' are congruent modulo Γ . Thus x and y are congruent modulo Γ .

We have shown that each fiber of γ is a Γ -class. The second assertion of the proposition follows by Proposition 2.2.1. \square

5. REMARKS ON ENUMERATION

A pleasant formula was obtained in [7] for the number of Baxter permutations in S_n :

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^n \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}.$$

This formula applies to twisted Baxter (i.e. 1-clumped) permutations and to diagonal rectangulations as well. In this section, we make several remarks on the problem

of enumerating generic rectangulations or 2-clumped permutations. In particular, we give some indications that the enumeration of 2-clumped permutations will be harder than the enumeration of 1-clumped permutations.

Remark 5.1. One way to enumerate generic rectangulations is by specializing a formula of Conant and Michaels [8]. This formula is a recursion, with signs, counting rectangulations according to the number of crosses. Thanks to Jim Conant for providing the results of his recursive calculations which verify and extend Table 1.

Remark 5.2. Another approach to enumerating 2-clumped permutations is to apply the key idea from [7]. This approach appears not to lead to a formula for the number of generic rectangulations, but is useful computationally, as we now explain.

Suppose $x \in G_n$. For each entry a in x , let $\beta(a) = \{b \in [a+1, n] : b \text{ is before } a\}$. Then $n+1$ can be placed before a in x to obtain another 2-clumped permutation if and only if one of the following holds: $\beta(a) = \emptyset$, $\beta(a) = [a+1, n]$, $\beta(a) = [a+1, c]$ for some c with $a+1 \leq c < n$, or $\beta(a) = [d, n]$ for some d with $a+1 < d \leq n$. Notice that if a satisfies none of these requirements, then even after $n+1$ is inserted elsewhere to obtain a permutation $x' \in G_{n+1}$, the entry a in x' still satisfies none of the requirements. Notice also that $n+1$ can be inserted after all of the entries of x to obtain a permutation in G_{n+1} .

Accordingly, we encode a 2-clumped permutation by a string of letters as follows. Read through the elements of x from left to right, and for each element a , write a letter in the string as follows:

- n** (for “null” or “ n ”) if $a = n$. Assume $a \neq n$ in the following cases.
- e** (for “empty”) if $\beta(a) = \emptyset$.
- f** (for “full”) if $\beta(a) = [a+1, n]$.
- l** (for “lower”) if $\beta(a) = [a+1, c]$ for some c with $a+1 \leq c < n$.
- u** (for “upper”) if $\beta(a) = [d, n]$ for some d with $a+1 < d \leq n$.

If none of these apply, then write nothing.

For example, for each of the permutations 2413, 4213, 3124 and 3142, the symbol 5 can be inserted anywhere except before the symbol 1. The sequences of letters for these permutations are respectively **en · f**, **nu · f**, **e · ln**, and **e · nf**, with a dot “ \cdot ” indicating an entry in the permutation that does not produce a letter. The respective strings are **enf**, **nuf**, **eln**, and again **enf**.

If we place the symbol $n+1$ before a in x or if we place $n+1$ after all entries of x , we can construct the string of letters corresponding to the new permutation x' by the following procedure. Insert the letter **n** in the string before the letter corresponding to a or at the end of the string and alter letters before the insertion according to the following rule: **n** becomes **e**, **e** is unchanged, **f** becomes **l**, **l** is unchanged, and **u** is deleted. Alter letters occurring after the insertion as follows: **n** becomes **f**, **e** becomes **u**, **f** is unchanged, **l** is deleted, and **u** is unchanged.

Now we can dispense with permutations entirely and simply insert letters into strings, counting the resulting strings by multiplicities. We start with the string **n**, encoding the permutation $1 \in G_1$. Inserting before or after the one letter in the string, we obtain **en** and **nf**, corresponding to the permutations $G_2 = \{12, 21\}$. Inserting into these two strings, we obtain the strings **een** (for 123), **enf** twice (for 132 and 231), **nuf** (for 312), **eln** (for 213), and **nff** (for 321). In the next round of insertions, deletions of letters come into play, so that for example, inserting **n** after the **e** in **eln**, we obtain **enf**. This corresponds to inserting 4 after 2 in 213 to obtain 2413. In all, there are 15 strings which represent the 24 permutations in G_4 .

The values shown in Table 1 are the results of a simple computer program that generates all strings and keeps track of multiplicities.

In contrast, representing 1-clumped permutations (i.e. twisted Baxter permutations) by strings leads to an enumeration formula. In this case the locations where $n + 1$ can be inserted are the locations labeled **n**, **e**, or **f**, with the same definitions as above. When **n** is inserted into the string, the remainder of the string is altered as follows: Before the insertion, **n** becomes **e**, **e** is unchanged, and **f** is deleted. After the insertion, **n** becomes **f**, **e** is deleted, and **f** is unchanged. All of the strings are of the form $\mathbf{e}^i \mathbf{n} \mathbf{f}^j$, for $i, j \geq 0$ and $i + j \leq n - 1$. Define $G(n, i, j)$ to be the multiplicity of the string $\mathbf{e}^i \mathbf{n} \mathbf{f}^j$ for 1-clumped permutations in S_n . Up to reindexing in n , the numbers $G(n, i, j)$ coincide with the numbers $T_n(i, j)$ in [7], and the obvious recurrence on $G(n, i, j)$ coincides with the recurrence on $T_n(i, j)$. This recurrence can be solved as in [7], or by the generating function method of [6]. In particular, the generating tree for the twisted Baxter permutations is isomorphic to the generating tree for Baxter permutations. Indeed, the original proof [18] that twisted Baxter permutations biject with Baxter permutations proceeded by establishing this isomorphism of generating trees.

Remark 5.3. Mallows [16] gave a combinatorial interpretation for the terms in formula for $B(n)$ by pointing out that the term indexed by k counts Baxter permutations with k *ascents* (or *rises*). There are two dual ways to define ascents: We will say that a *right ascent* is a pair of adjacent entries such that the left entry in the pair is smaller than the right entry in the pair. A *left ascent* is a pair of entries i and $i - 1$ with $i - 1$ appearing before i in the permutation. We can similarly define *right descents* (left entry in the pair larger) and *left descents* ($i - 1$ appearing after i). Recall that the Baxter permutations are the permutations avoiding 3-14-2 and 2-41-3. It is easy to see that a given permutation is a Baxter permutation if and only if its inverse is a Baxter permutation. (See e.g. [15, Corollary 4.2].) Thus, when counting Baxter permutations according to the number of ascents, it does not matter whether we use right ascents or left ascents. Furthermore, it is immediate that a permutation is a Baxter permutation if and only if its reverse permutation is also a Baxter permutation. Thus the formula for Baxter permutations with a fixed number of ascents is the same as the formula for Baxter permutations with a fixed number of descents. It is easy to see that the number of ascents in a permutation x equals the number of vertical walls in the diagonal rectangulation $\rho(x)$. Thus the formula for $B(n)$ counts diagonal rectangulations according to the number of vertical walls.

The number of left ascents of x also equals the number of vertical walls in the generic rectangulation $\gamma(x)$ and the number of left descents of x equals the number of horizontal walls. Thus by the symmetry of the rectangulations, counting 2-clumped permutations by left descents is equivalent to counting 2-clumped permutations by left ascents. However, the inverse of a 2-clumped permutation is not necessarily a 2-clumped permutation, so it matters whether we take the left or right definitions of descents or ascents. Thus there are at least three reasonable statistics by which to count: left ascents/descents, right ascents, or right descents. Computations show that these three statistics are distributed differently, and suggest the following conjecture:

Conjecture 5.4. Fix $k \geq 0$. Then for $n \geq 1$, the number of 2-clumped permutations in S_n with exactly d right descents is a polynomial $p_k(n)$ of degree $3d$ and leading

coefficient

$$\prod_{i=1}^d \frac{2}{i(i+1)(i+2)} = \frac{2^{d+1}}{d!(d+1)!(d+2)!}.$$

The polynomial $p_k(n)$ must have factors $(n-1)(n-2)\cdots(n-d)$, so the point is to determine the polynomial $\tilde{p}_k(n)$ of degree $2d$ that results when these factors are taken out. The first few polynomials appear to be $\tilde{p}_0(n) = 1$,

$$\tilde{p}_1(n) = (n^2 - 2n + 3)/3,$$

$$\tilde{p}_2(n) = (5n^4 - 36n^3 + 142n^2 - 279n + 270)/180, \text{ and}$$

$$\tilde{p}_3(n) = (14n^6 - 213n^5 + 1688n^4 - 8361n^3 + 26000n^2 - 46884n + 37800)/15120.$$

It should be emphasized that the point of the conjecture is to find a formula enumerating all 2-clumped permutations. The conjecture can be proved for some small values of k , and proofs for additional values of k are of interest only to the extent to which they lead to a conjecture on the general form of $p_k(n)$.

The other two statistics (left ascents/descents and right ascents) do not lead to polynomial formulas. In particular, counting 2-clumped permutations by left ascents, or equivalently counting generic rectangulations by the number of vertical walls, appears to be hard.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC, USA